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Generalized Convexity and Nonsmooth Problems of Vector Optimization

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Abstract. In this paper it is shown that every generalized Kuhn–Tucker point of a vector optimization problem involving locally Lipschitz functions is a weakly efficient point if and only if this problem is KT-pseudoinvex in a suitable sense. Under a closedness assumption (in particular, under a regularity condition of the constraint functions) it is pointed out that in this result the notion of generalized Kuhn–Tucker point can be replaced by the usual notion of Kuhn–Tucker point. Some earlier results in (Martin (1985), *Journal of Optimization Theory and Applications* 47, 65–76; Osuna-Gómez et al. (1999), *Journal of Mathematical Analysis and Applications* 233, 205–220; Osuna-Gómez et al. (1998), *Journal of Optimization Theory and Applications* 98, 651–661, Phuong et al. (1995), *Journal of Optimization Theory and Applications* 87, 579–594) are included as special cases of ours. The paper also contains characterizations of HC-invexity and KT-invexity properties which are sufficient conditions for KT-pseudoinvexity property of nonsmooth problems.

Mathematics Subject Classifications: 90C29; 26B25

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1. Introduction

Let $\bar{f}$ and g_i $(i = 1, 2, \ldots, m)$ be differentiable functions defined on an Euclidean space R^n. Consider the following *Mathematical Programming Problem* (P)

$$\min \quad \bar{f}(x) \tag{1.1}$$

$$\text{subject to} \quad x \in S := \{u : g_i(u) \leqq 0 \quad (i = 1, 2, \ldots, m)\}. \tag{1.2}$$

Here and in the sequel the notation $a \leqq b$ for real numbers a and b means that a is less than or equal to b. Let $x_0 \in S$ and

$$I_0 := I(x_0) = \{i : g_i(x_0) = 0\}. \tag{1.3}$$

Hanson [6] was the first who showed that under a generalized convexity requirement, later called invexity, every Kuhn–Tucker point is a minimizer of (P). Recall [6] that $\bar{f}$ and g_i $(i \in I_0)$ are invex on S at x_0 if there is a map $\eta : S \to R^n$ such that for all $x \in S$

$$\bar{f}(x) - \bar{f}(x_0) \geqq \bar{f}_{x_0}'(\eta(x)),$$
$$g_i(x) - g_i(x_0) \geqq g_{ix_0}'(\eta(x)) \quad (i \in I_0),$$

where $\bar{f}_{x_0}'$ and g_{ix_0}' denote the Fréchet derivatives of $\bar{f}$ and g_i at x_0, respectively. In this case, Martin [9] said that Problem (P) is HC-invex on S at x_0. Thus, every Kuhn–Tucker point is a minimizer if Problem (P) is HC-invex at this point. Martin [9] remarked that the converse is not true in general, and he proposed a weaker notion, called KT-invexity, which assures that every Kuhn–Tucker point is a minimizer of Problem (P) if and only if Problem (P) is KT-invex. In [11, 12] this result was established for a vector optimization problem with differentiable data. In this paper, we shall extend this property to a vector optimization problem involving *locally Lipschitz functions*. The case where S is an arbitrary subset which may not be given by inequality constraints (1.2) is also considered. Our obtained extensions are useful since many problems often encountered in economics, engineering design ... can be described only by locally Lipschitz functions (see [3]). The main tools we use are nonsmooth versions of the corresponding HC-invexity, KT-invexity and KT-pseudoinvexity notions for differentiable programs [6, 9, 11, 12]. Nonsmooth invexity is introduced by Craven [4] and is used in [15] to prove the converse Kuhn–Tucker condition for locally Lipschitz programs. Observe [7, 8] that the invexity with *nontrivial kernel* η for the objective function and the constraint functions can be seen as a necessary optimality condition for Problem (P) with differentiable data.

In this paper, we consider *a generalized Kuhn–Tucker point* of a vector optimization problem involving locally Lipschitz functions, weakly efficient solutions of the problem and KT-pseudoinvexity of the problem, and show that the generalized Kuhn–Tucker point of the problem is a weakly efficient solution if and only if the problem is KT-pseudoinvex. Under a closedness assumption, it is shown that in the just mentioned result the notion of generalized Kuhn–Tucker point can be replaced by the usual notion of Kuhn–Tucker point. The KT-pseudoinvexity is a generalization of HC-invexity and KT-invexity. We give stronger characterizations for the problem to be KT-invex or HC-invex by using the scalarization of its objective functions or its Lagrange functions. Furthermore, we define a generalized stationary point of a vector optimization problem with a "geometric" constraint set by using the Clarke normal cone, and show that the generalized stationary point of the problem is a weakly efficient solution if and only if the problem is KT-pseudoinvex.

The organization of this paper is as follows. Section 2 gives definitions of HC-invexity, KT-invexity and KT-pseudoinvexity properties of a nonsmooth problem of vector optimization, and shows that the last two invexity properties coincide in the case of scalar optimization. Section 3 recalls some criteria of consistency of convex inequalities which are needed in the subsequent sections. Section 4 contains extensions of results of [9, 11, 12] to a vector optimization problem where S is given by a system of nonsmooth inequalities. Section 5 considers a problem with a "geometric" constraint set S.

2. Preliminaries

Let A be an arbitrary nonempty subset of an Euclidean space R^n. We say that A is a convex cone if $\lambda a \in A$ for all $\lambda \geqq 0$, $a \in A$ and if $\alpha a_1 + (1-\alpha)a_2 \in A$ for all $a_1 \in A$, $a_2 \in A$ and $\alpha \in [0,1]$. We denote by co A (resp. cl A) the convex hull (resp. the closure) of A. The cone generated by A is denoted by cone A:

$$\text{cone } A := \{\lambda a : \lambda \geqq 0, \ a \in A\}.$$

For simplicity of notation we write cl cone A and cl cone co A instead of cl (cone$A$) and cl [cone (co A)] respectively.

Let $d_A(x)$ be the distance from x to $A \subset R^n$. Given a sequence of subsets $A_l \subset R^n$ $(l = 1, 2, \dots)$ we write $x \in \lim \inf_{l\to\infty} A_l$ if $d_{A_l}(x) \to 0$ as $l \to \infty$. If A_l $(l = 1, 2, \dots)$ are real numbers then $x \in \lim \inf_{l\to\infty} A_l$ means that $|A_l - x| \to 0$ $(l \to \infty)$. Thus, for a sequence of real numbers A_l if $\lim \inf_{l\to\infty} A_l$ is understood in the usual sense then $\lim \inf_{l\to\infty} A_l$ may be an empty set while $\lim \inf_{l\to\infty} A_l$ does exist.

For vectors $x = (x_1, x_2, \dots, x_n) \in R^n$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in R^n$ we write $x = \xi$ (resp. $x \leqq \xi$; $x < \xi$) if the equalities $x_i = \xi_i$ (resp. the inequalities $x_i \leqq \xi_i$; $x_i < \xi_i$) hold for all $i = 1, 2, \dots, n$. We write $x \leqslant \xi$ if the inequalities $x_i \leqq \xi_i$ hold for all $i = 1, 2, \dots, n$, and if at least one of these inequalities is strict.

We denote by $\langle x, \xi \rangle$ the scalar product of x and ξ: $\langle x, \xi \rangle = \sum_{i=1}^n x_i \xi_i$. The symbol $\langle x, A \rangle$ is defined as the set $\{\langle x, \xi \rangle : \xi \in A\}$.

Let $f = (f_1, f_2, \dots, f_p)$ and $g = (g_1, g_2, \dots, g_m)$ be locally Lipschitz vector-valued maps defined on R^n. We write

$$f^0(x_0, \cdot) = (f_1^0(x_0, \cdot), f_2^0(x_0, \cdot), \dots, f_p^0(x_0, \cdot)),$$

$$\partial f(x_0) = \partial f_1(x_0) \times \partial f_2(x_0) \times \dots \times \partial f_p(x_0),$$

where $f_j^0(x_0, \cdot)$ and $\partial f_j(x_0)$, introduced in [3], are Clarke directional derivative and subdifferential of f_j at x_0. We denote by $T_A(x_0)$ and $N_A(x_0)$ the Clarke tangent cone and the Clarke normal cone of $A \subset R^n$ at $x_0 \in A$. Recall [3] that

$$T_A(x_0) = \{x : d_A^0(x_0, x) = 0\}, \tag{2.1}$$

$$N_A(x_0) = \{\xi : \langle \xi, x \rangle \leqq 0 \quad \forall x \in T_A(x_0)\} = \text{cl cone } \partial d_A(x_0), \tag{2.2}$$

where $d_A^0(x_0;\cdot)$ and $\partial d_A(x_0)$ stand for the Clarke directional derivative and Clarke subdifferential of the distance function $d_A(\cdot)$ at x_0.

Consider the following *Vector Optimization Problem* (VOP)

$$\min \quad f(x) \tag{2.3}$$

$$\text{subject to} \quad g(x) \leqq 0. \tag{2.4}$$

Let $S = \{x : g(x) \leqq 0\}$ i.e. $S = \{x : g_i(x) \leqq 0 \ (i = 1, 2, \ldots, m)\}$. A point $x_0 \in S$ is called a *weakly efficient point* of Problem (VOP) if for any point $x \in S$ the following condition does not hold:

$$f(x) < f(x_0). \tag{2.5}$$

Necessary condition for $x_0 \in S$ to be a weakly efficient point is established in [3]. Under some regularity assumption a weakly efficient point must be a Kuhn–Tucker point. Before formulating the definition of a Kuhn–Tucker point let us denote by $(\mu_j)_{j\in J}$ a vector with components μ_j $(j \in J)$ and $(\lambda_i)_{i\in I_0}$ a vector with components λ_i $(i \in I_0)$ where $J = \{1, 2, \ldots, p\}$ and I_0 is defined by (1.3).

DEFINITION 2.1. A point $x_0 \in S$ is a Kuhn–Tucker point of Problem (VOP) if there are vectors $\mu := (\mu_j)_{j\in J} \geqslant 0$ and $\lambda := (\lambda_i)_{i\in I_0} \geqq 0$ such that

$$0 \in H(\mu, \lambda, x_0), \tag{2.6}$$

where

$$H(\mu, \lambda, x_0) = \sum_{j\in J} \mu_j \, \partial f_j(x_0) + \sum_{i\in I_0} \lambda_i \, \partial g_i(x_0). \tag{2.7}$$

Observe that (2.6) means that there are points $c_j \in \partial f_j(x_0)$ $(j \in J)$ and $b_i \in \partial g_i(x_0)$ $(i \in I_0)$ such that

$$0 = \sum_{j\in J} \mu_j c_j + \sum_{i\in I_0} \lambda_i b_i.$$

Let us introduce the following **condition** (CQ) at $x_0 \in S$: if $I_0 \neq \emptyset$ then

$$0 \notin \text{ co } \bigcup_{i\in I_0} \partial g_i(x_0). \tag{2.8}$$

From Theorem 6.1.3 of [3] it follows that if x_0 is a weakly efficient point of (VOP) then under condition (CQ) at least one of the Lagrange multipliers associated to the objective functions of (VOP) is nonzero.

DEFINITION 2.2. Problem (VOP) is HC-invex on S at x_0 if f_j $(j \in J)$ and g_i $(i \in I_0)$ are invex on S at x_0 in the sense of Craven [4]: there is a map $\eta : S \to R^n$ such that for all $x \in S$

$$f_j(x) - f_j(x_0) \ \geqq f_j^0(x_0, \eta(x)) \quad (j \in J), \tag{2.9}$$

$$g_i(x) - g_i(x_0) \ \geqq \ g_i^0(x_0, \eta(x)) \quad (i \in I_0). \tag{2.10}$$

DEFINITION 2.3. Problem (VOP) is KT-invex on S at x_0 if there is a map $\eta : S \to R^n$ such that for all $x \in S$

$$f_j(x) - f_j(x_0) \geqq f_j^0(x_0, \eta(x)) \quad (j \in J), \tag{2.11}$$

$$-g_i(x_0) \geqq g_i^0(x_0, \eta(x)) \quad (i \in I_0). \tag{2.12}$$

DEFINITION 2.4. Problem (VOP) is KT-pseudoinvex on S at x_0 if there is a map $\eta : S \to R^n$ such that for all $x \in S$

$$f(x) < f(x_0) \quad \Rightarrow \quad \begin{cases} 0 > f_j^0(x_0, (x)) \quad (j \in J), & (2.13) \\ 0 \geqq g_i^0(x_0, \eta(x)) \quad (i \in I_0). & (2.14) \end{cases}$$

DEFINITION 2.5. Problem (VOP) is KT-pseudoinvex if for all $x_0 \in S$ it is KT-pseudoinvex on S at x_0. Similarly for HC-invexity and KT-invexity properties of Problem (VOP).

Relationships between the above notions of invexity are given by

PROPOSITION 2.1.

1. *For any Problem (VOP) and any point $x_0 \in S$,* *HC-invexity on S at $x_0 \Rightarrow$ KT-invexity on S at $x_0 \Rightarrow$ KT-pseudoinvexity on S at x_0.*
2. *For any Problem* (P) (*i.e. Problem (VOP) with $p = 1$*) *and any point $x_0 \in S$, KT-invexity on S at x_0* $\Leftrightarrow$ KT-*pseudoinvexity on S* at x_0.

Proof. The first part of Proposition 2.1 is obvious from the very definitions. To prove the second one it is enough to show that for the case $p = 1$ the following implication is true:

$$\text{KT-pseudoinvexity on } S \text{ at } x_0 \Rightarrow \text{ KT-invexity on } S \text{ at } x_0. \tag{2.15}$$

Indeed, let $x \in S$. If $f(x) \geqq f(x_0)$ then (2.11) and (2.12) are satisfied, with $\eta(x) = 0$. If $f(x) < f(x_0)$ then by assumption there is a point $\eta(x)$ satisfying (2.13) and (2.14). Since $\gamma f^0(x_0, \eta(x)) \to -\infty$ as $\gamma \to +\infty$, we can take $\gamma > 0$ such that

$$f(x) - f(x_0) \geqq \gamma f^0(x_0, \eta(x)) = f^0(x_0, \gamma\eta(x)).$$

On the other hand, we have from (2.14)

$$-g_i(x_0) = 0 \geqq \gamma g_i^0(x_0, \eta(x)) = g_i^0(x_0, \gamma\eta(x)) \ (i \in I_0).$$

Therefore, (2.11) and (2.12) are satisfied, with $\gamma\eta(x)$ instead of $\eta(x)$. Implication (2.15) is thus established. □

REMARK 2.1. Implication (2.15) is no longer true in case $p > 1$. In other words, the equivalence formulated in the second part of Proposition 2.1 fails to hold if $p > 1$. This is shown by Example 4.3 of Section 4 (see also Example 3.5 of [17]).

In this paper Problem (VOP) is called differentiable if all functions f_j and g_i are Fréchet differentiable. (Observe that in such a problem functions f_j and g_i may not be locally Lipschitz, except for the case where they are continuously Fréchet differentiable.) When dealing with differentiable problems we shall use the Fréchet derivatives f'_{jx_0} and g'_{ix_0} of f_j and g_i at x_0 instead of the Clarke subdifferentials $\partial f_j(x_0)$ and $\partial g_i(x_0)$. We also observe that in this case, for all $x \in R^n$, we shall use $f'_{jx_0}(x)$ and $g'_{ix_0}(x)$ in place of $f_j^0(x_0, x)$ and $g_i^0(x_0, x)$. Thus, for the differentiable case Definitions 2.2 and 2.4 reduce to the known properties of HC-invexity [9] and KT-pseudoinvexity [11]. Observe that even in the differentiable case KT-invexity in Definition 2.3 does not coincide with KT-invexity defined in [11] since in [11] (2.11) is replaced by $f(x) - f(x_0) \geqslant f'_{x_0}(\eta(x))$. The KT-invexity in [12] is renamed as KT-pseudoinvexity in [11].

3. Consistency of Convex Inequalities

In this section we give two propositions on the consistency of systems of convex inequalities which will be used in the subsequent sections.

Let

$$J = \{1, 2, \ldots, p\},$$
$$I = \{1, 2, \ldots, k\}.$$

Given nonempty compact convex sets C_j $(j \in J)$ and B_i $(i \in I)$ in R^n, we define

$$\psi_j(\xi) = \max_{c \in C_j} \langle c, \xi \rangle,$$
$$\varphi_i(\xi) = \max_{b \in B_i} \langle b, \xi \rangle$$

and consider the consistency of the following system of inequalities of variable $\xi \in R^n$

$$\psi(\xi) < 0, \tag{3.1}$$

$$\varphi(\xi) \leqq 0, \tag{3.2}$$

where ψ (resp. φ) denotes the vector with components ψ_j (resp. φ_i).

PROPOSITION 3.1. *System* (3.1), (3.2) *has a solution if and only if*

$$0 \notin \text{co} \bigcup_{j\in J} C_j + \text{cl cone co} \bigcup_{i\in I} B_i. \tag{3.3}$$

Proof. We see that

$$(3.3) \iff [-\text{co} \bigcup_{j\in J} C_j] \cap \text{cl cone co} \bigcup_{i\in I} B_i = \emptyset$$

$\iff$ (by a separation theorem) $\exists \xi \in R^n$ such that $\langle v, \xi \rangle < 0$ and $\langle w, \xi \rangle \leqq 0$ for all $v \in \text{co} \bigcup_{j\in J} C_j$ and $w \in \text{cl cone co} \bigcup_{i\in I} B_i$.

Thus the conclusion of Proposition 3.1 is true. □

Now let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_p)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_k)$. Consider the following nonhomogeneous system of inequalities of variable $\xi \in R^n$

$$\psi(\xi) \leqq \alpha, \tag{3.4}$$

$$\varphi(\xi) \leqq \beta. \tag{3.5}$$

Let

$$C_j' = C_j \times \{-\alpha_j\} \subset R^n \times R,$$

$$B_i' = B_i \times \{-\beta_i\} \subset R^n \times R.$$

The following result is an easy consequence of Proposition 3.1.

PROPOSITION 3.2. *System* (3.4), (3.5) *has a solution if and only if*

$$(0, 1) \notin \text{cl cone co} \left\{ \bigcup_{j\in J} C_j', \bigcup_{i\in I} B_i' \right\}$$

where 0 *denotes the origin of* R^n.

In the next section, for every point $x_0 \in S$ we set $I = I(x_0) = I_0$, $C_j = \partial f_j(x_0)$ and $B_i = \partial g_i(x_0)$. Hence by [3] $\psi_j(\xi) = f_j^0(x_0, \xi)$ and $\varphi_i(\xi) = g_i^0(x_0, \xi)$ $(\xi \in R^n)$.

4. Vector Optimization Problem with Inequality Constraints

In this section, unless otherwise specified we shall assume that f and g are locally Lipschitz vector-valued maps and S is given by (1.2).

Let $x_0 \in S$ and $t \in R$. Consider the following subsets of $R^n \times R$:

$$D_1(x_0, t) = \bigcup_{j\in J} [\partial f_j(x_0) \times \{t\}],$$

$$D_2(x_0, 0) = \bigcup_{i\in I_0} [\partial g_i(x_0) \times \{0\}],$$

$$D(x_0, t) = D_1(x_0, t) \bigcup D_2(x_0, 0).$$

Recall that $I_0 = I(x_0)$ is defined by (1.3). We begin by a technical lemma.

LEMMA 4.1. *The following statements are equivalent*:

(a) *For some* $t \neq 0$ cone co $D(x_0, t)$ *is closed.*
(b) *For all* $t' \neq 0$ cone co $D(x_0, t')$ *is closed.*

Proof. Obviously (b) $\Rightarrow$ (a). To prove the converse implication let us observe that

$$(\xi, r) \in \text{cone co } D(x_0, t') \iff (\xi, tt'^{-1}r) \in \text{cone co } D(x_0, t),$$

$$(\xi, r) \in \text{cl cone co } D(x_0, t') \iff (\xi, tt'^{-1}r) \in \text{cl cone co } D(x_0, t).$$

On the other hand, by assumption (a)

$$\text{cone co } D(x_0, t) = \text{cl cone co } D(x_0, t).$$

Therefore, $(\xi, r) \in \text{cl cone co } D(x_0, t') \iff (\xi, r) \in \text{cone co } D(x_0, t')$.
Thus cone co $D(x_0, t')$ is closed. □

A sufficient condition for the closedness of cone co $D(x_0, t)$ is given by

LEMMA 4.2. *If condition (CQ) holds then for all* $t \neq 0$ cone co $D(x_0, t)$ *is closed.*

Proof. Our desired conclusion is proved if $0 \notin$ co $D(x_0, t)$ since in this case cone co $D(x_0, t)$ is a closed set (see [16, Corollary 9.6.1]). Assume to the contrary that $0 \in$ co $D(x_0, t)$. Then there are nonnegative numbers μ_j $(j \in J)$ and λ_i $(i \in I_0)$ such that

$$1 = \sum_{j \in J} \mu_j + \sum_{i \in I_0} \lambda_i, \tag{4.1}$$

$$0 \in \sum_{j \in J} \mu_j \partial f_j(x_0) + \sum_{i \in I_0} \lambda_i \partial g_i(x_0), \tag{4.2}$$

$$0 = \sum_{j \in J} \mu_j t + \sum_{i \in I_0} \lambda_i .0. \tag{4.3}$$

Since $t \neq 0$ (4.3) yields $\mu_j = 0$ $(j \in J)$. Hence (4.1) and (4.2) contradict condition (CQ) (see (2.8)). □

REMARK 4.1. If instead of the Lipschitz property of f_j and g_i we assume that they are Fréchet differentiable at x_0 and if in the definition of $D_1(x_0, t)$ and $D_2(x_0, 0)$ we replace the subdifferentials $\partial f_j(x_0)$ and $\partial g_i(x_0)$ by the Fréchet derivatives f'_{jx_0} and g'_{ix_0} then the closedness of cone co $D(x_0, t)$ is automatically satisfied for all $t \in R$. This follows from the fact that the convex cone generated by a finitely many points must be closed (see [16, Theorem 19.1]). Thus for differentiable problem (VOP) the sets cone co $D(x_0, t)$ are always closed. The same is true for the sets cone co $M(x_0, x)$ and cone co $Q(x_0, x)$ to be introduced later.

REMARK 4.2. Condition (CQ) is not a necessary condition for the closedness of cone $D(x_0, t)$.

EXAMPLE 4.1. Consider Problem (VOP) with $n = p = m = 1$. For $x \in R$, $f_1(x) = x$ and $g_1(x) = |x|$. For $x_0 = 0$ we have $\partial f_1(x_0) = \{1\}$, $I_0 = \{1\}$ and $\partial g_1(x_0) = [-1, +1]$. Obviously, condition (CQ) is violated, but cone co $D(x_0, 1)$ is closed (and hence, by Lemma 3.1 cone co $D(x_0, t)$ is closed for all $t \neq 0$).

The following example proves that Lemma 4.2 may fail to hold without condition (CQ).

EXAMPLE 4.2. Consider Problem (VOP) with $n = 2$, $p = m = 1$. For $x = (x_1, x_2) \in R^2$ let us set $f_1(x) = x_1$ and $g_1(x) = (x_1^2 + x_2^2)^{1/2} + x_2$. Since in our case $p = 1$, (VOP) becomes Problem (P) and hence, by Proposition 2.1, (VOP) is KT-pseudoinvex if and only if it is KT-invex. Setting $x_0 = (0, 0) \in R^2$ and using the easily checked equalities $\partial f_1(x_0) = \{(1, 0)\}$ and $\partial g_1(x_0) = \{(x_1, x_2) \in R^2 : x_1^2 + (x_2 - 1)^2 \leqq 1\}$ we see that condition (CQ) does not hold at x_0 (since $0 \in \partial g_1(x_0)$) and cone co $D(x_0, t)$ is not closed for all $t \neq 0$.

DEFINITION 4.1. A point $x_0 \in S$ is a generalized Kuhn–Tucker point of Problem (VOP) if there is a vector $\mu := (\mu_j)_{j \in J} \geqslant 0$ such that

$$0 \in \sum_{j \in J} \mu_j \partial f_j(x_0) + \text{cl cone co} \bigcup_{i \in I_0} \partial g_i(x_0). \tag{4.4}$$

Obviously, a Kuhn–Tucker point is also a generalized Kuhn–Tucker point and the converse is true if the set cone co $\bigcup_{i \in I_0} \partial g_i(x_0)$ is closed.

DEFINITION 4.2. We say that the closedness assumption is satisfied at $x_0 \in S$ if there is $t \neq 0$ such that cone co $D(x_0, t)$ is closed. If this is true for all $x_0 \in S$ then we simply say that the closedness assumption is satisfied.

Let us observe by Lemma 4.2 that condition (CQ) implies the closedness assumption at x_0.

In [18] Treiman introduced a qualification condition for closed subsets $A_i \subset R^n$ $(i = 1, 2, \ldots, m')$ at $x_0 \in \bigcap_{i=1}^{m'} A_i$ by requiring that the set-valued map

$$v = (v_1, v_2, \ldots, v_{m'}) \in R^{m'} \mapsto \Phi(v) = \bigcap_{i=1}^{m'} (A_i + v_i)$$

is pseudo-Lipschitz [1] at $(0, x_0)$. We shall prove that this Treiman condition, applied to the sets

$$A_i = \{x : g_i(x) \leqq 0\} \quad (i = 1, 2, \ldots, m'),$$

is more general than our condition (CQ); however, unlike condition (CQ), it does not assure the validity of the closedness assumption at x_0. Indeed, it is known from [18, Prop. 3.3] that the Treiman qualification condition at x_0 is characterized by the following condition:

$$\left[a_i \in \widetilde{N}_{A_i}(x_0) \ (i = 1, 2, \ldots, m'), \ \sum_{i=1}^{m'} a_i = 0 \right] \Rightarrow a_i = 0 \quad (i = 1, 2, \ldots, m'),$$

where $\widetilde{N}_{A_i}(x_0)$ denotes the Mordukhovich normal cone of A_i at x_0 (see [10]).

Without loss of generality we may assume that $I_0 = I(x_0) = \{1, 2, \ldots, m'\}$ $(m' \leqq m)$. Then condition (CQ) implies that $0 \notin \partial g_i(x_0)$ $(i = 1, 2, \ldots, m')$; hence, by Corollary 1 of Theorem 2.4.7 of [3]

$$N_{A_i}(x_0) \subset \text{cone}\, \partial g_i(x_0).$$

Since by [10] the Mordukhovich normal cone is a subset of the Clarke normal cone, we derive from the just written inclusion that, for each $a_i \in \widetilde{N}_{A_i}(x_0)$, there exists $\lambda_i \geqq 0$ with $a_i \in \lambda_i \partial g_i(x_0)$. Using this fact we easily check that (2.8) implies the above characterization of the Treiman qualification condition. Thus

$$\text{condition (CQ) at } x_0 \Rightarrow \text{Treiman qualification condition at } x_0.$$

The converse of this implication is no longer true. To see this, assume that $m = 1$, i.e. g is a scalar function. Then the Treiman qualification condition holds automatically (see [18, p. 1320]), while our condition (CQ) requires that $0 \notin \partial g(x_0)$ (i.e. condition (CQ) holds not automatically). Example 4.2 illustrates that the Treiman qualification condition holds (since $m = 1$) but it does not imply the closedness assumption at x_0. This example also proves that the constraint qualification given in [2, p. 2424] is not sufficient for the validity of our closedness assumption.

To conclude the discussion on links between condition (CQ) and constraint qualification conditions of [2, 18], we restrict ourselves to optimization problems with inequality constraints only, and we observe that usually a constraint qualification is understood as a condition under which the

Lagrange multipliers associated to the objective functions of (scalar or) multiobjective optimization problems are not all zero. In this classical sense, condition (CQ) is a constraint qualification, while the above mentioned conditions of both [2, 18] are not.

Let us mention the following result showing that under the closedness assumption the notions of Kuhn–Tucker points and generalized Kuhn–Tucker points are equivalent.

LEMMA 4.3. *Let the closedness assumption be satisfied at x_0. Then x_0 is a Kuhn–Tucker point if and only if it is a generalized Kuhn–Tucker point.*

Proof. It is enough to establish the sufficiency part of Lemma 4.3. Indeed, if x_0 is a generalized Kuhn–Tucker point then there are points $c_j \in \partial f_j(x_0)$ $(j \in J)$, vector $(\mu_j)_{j\in J} \geqslant 0$ and sequences $(\lambda_i^l)_{i\in I_0} \geqslant 0$, $b_i^l \in \partial g_i(x_0)$ $(l = 1, 2, \ldots)$ such that

$$\sum_{j\in J} \mu_j c_j + \sum_{i\in I_0} \lambda_i^l b_i^l \to 0$$

as $l \to \infty$. Setting $t = \sum_{j\in J} \mu_j > 0$, we derive that

$$\sum_{j\in J} \mu_j (c_j, 1) + \sum_{i\in I_0} \lambda_i^l (b_i^l, 0) \to (0, t),$$

i.e. $(0, t) \in \text{cl cone co } D(x_0, t)$. By Lemma 4.1 the closedness assumption at x_0 implies the closedness of $\text{cone co } D(x_0, t')$ for all $t' \neq 0$. Since $t \neq 0$, $\text{cone co } D(x_0, t)$ is closed. Thus $(0, t) \in \text{cone co } D(x_0, t)$, proving that x_0 is a Kuhn–Tucker point. □

We are now in a position to formulate

THEOREM 4.1. *Consider the following statements:*

(a) *Problem (VOP) is KT-pseudoinvex on S at x_0.*
(b) *If x_0 is a generalized Kuhn–Tucker point then x_0 is a weakly efficient point.*
(c) *If x_0 is a Kuhn–Tucker point then x_0 is a weakly efficient point.*

Then (a) $\Leftrightarrow$ (b). *If, in addition, the closedness assumption is satisfied at x_0 (in particular, if condition (CQ) holds at x_0) then all three above statements are equivalent.*

Proof. It is enough to prove the first conclusion of Theorem 4.1. The second conclusion is a consequence of the first one and Lemma 4.3.

(a) $\Rightarrow$ (b). Assume to the contrary that there is a vector $\mu = (\mu_j)_{j\in J} \geq 0$ such that (4.4) is satisfied, but x_0 is not a weakly efficient point i.e. (2.5) is satisfied for some $x \in S$. By the KT-pseudoinvexity property there is $\eta(x)$ satisfying (2.13) and (2.14). On the other hand, (4.4) shows that there are points $c_j \in \partial f_j(x_0)$ and sequences $(\lambda_i^l)_{i \in I_0} \geqq 0$, $b_i^l \in \partial g_i(x_0)$ $(l = 1, 2, \ldots)$ such that

$$-\sum_{j \in J} \mu_j c_j = \lim_{l \to \infty} \sum_{i \in I_0} \lambda_i^l b_i^l. \tag{4.5}$$

Making use of (2.13) and (2.14) we obtain

$$0 > \sum_{j \in J} \mu_j \langle c_j, \eta(x) \rangle \ , \tag{4.6}$$

$$0 \geqq \sum_{i \in I_0} \lambda_i^l \langle b_i^l, \eta(x) \rangle \quad (l = 1, 2, \ldots). \tag{4.7}$$

Letting $l \to \infty$ in (4.7) and taking (4.5) into account we get

$$0 \geqq -\sum_{j \in J} \mu_j \langle c_j, \eta(x) \rangle \ ,$$

a contradiction to (4.6).

(b) $\Rightarrow$ (a) If

$$0 \in \mathrm{co} \bigcup_{j \in J} \partial f_j(x_0) + \mathrm{cl\ cone\ co} \bigcup_{i \in I_0} \partial g_i(x_0) \tag{4.8}$$

then by statement (b) there is no $x \in S$ satisfying (2.5). Thus in this case the KT-pseudoinvexity property is satisfied. If (4.8) does not hold then by Proposition 3.1 system

$$f_j^0(x_0, \xi) < 0 \quad (j \in J), \tag{4.9}$$

$$g_i^0(x_0, \xi) \leqq 0 \quad (i \in I_0) \tag{4.10}$$

has a solution ξ. If we set $\eta(x) = \xi$ for all x satisfying (2.5) then (2.13) and (2.14) hold. □

REMARK 4.3. We have seen in the proof of Theorem 4.1 that the second conclusion of this theorem is obtained by combining the first one and Lemma 4.3. It is worth noticing that this result can be derived by using (the first conclusion of Theorem 4.1 and) Proposition 3.2 instead of Lemma 4.3. Indeed, since the implication (b) $\Rightarrow$ (c) is obvious, it is enough to show that (c) $\Rightarrow$ (a). Assume to the contrary that (2.5) holds for some $x \in S$ but system (4.9), (4.10) has no solution. Observe that the inconsistency of system (4.9), (4.10) implies the inconsistency of system

$$f_j^0(x_0, \xi) \leqq t' \quad (j \in J),$$

$$g_i^0(x_0, \xi) \leqq 0 \quad (i \in I_0),$$

where

$$t' := \min_{j\in J} \left[f_j(x) - f_j(x_0) \right] < 0.$$

Hence by Proposition 3.2 and Lemma 4.1

$$(0, 1) \in \text{cl cone co } D(x_0, -t') = \text{cone co } D(x_0, -t').$$

Thus there are vectors $\mu \geqq 0$ and $\lambda \geqq 0$ such that (2.6) is satisfied and

$$1 = -\sum_{j\in J} \mu_j . t' - \sum_{i\in I_0} \lambda_i . 0.$$

Since $t' < 0$ the last equality shows that $\mu \neq 0$. By statement (c) x_0 is a weakly efficient point. This contradicts (2.5).

We now give some examples.

EXAMPLE 4.3. Consider Problem (VOP) with $n = p = 2$ and $m = 1$. For $x = (x_1, x_2) \in R^2$ define $f_1(x) = x_1 + x_1x_2$, $f_2(x) = x_1 + x_2 - x_1x_2$ and $g_1(x) = (x_1^2 + x_2^2)^{1/2} + x_2$. It is easily seen that $S = \{(x_1, x_2) \in R^2$: $x_1 = 0,\ x_2 \leqq 0\}$ and $\text{co}[\bigcup_{i=1,2} \partial f_i(x_0)] = \{(x_1, x_2) \in R^2$: $x_1 = 1,\ x_2 \in [0, 1]\}$ where $x_0 : = (0, 0) \in R^2$. Making use of the formulae of $\partial g_1(x_0)$ given in Example 4.2 we can check that

$$-\text{co}\left[\bigcup_{i=1,2} \partial f_i(x_0)\right] \bigcap \text{cone } \partial g_1(x_0) = \emptyset,$$

$$-\text{co}\left[\bigcup_{i=1,2} \partial f_i(x_0)\right] \bigcap \text{cl cone } \partial g_1(x_0) = \{(-1, 0)\}.$$

The first of these conditions shows that x_0 is not a Kuhn–Tucker point, and the second proves that x_0 is a *generalized* Kuhn–Tucker point. It is easy to see that x_0 is a weakly efficient point and (VOP) is KT-pseudoinvex at x_0. Observe that (VOP) is not KT-invex at x_0. Indeed, for $x = (0, -1) \in S$ we cannot find $\eta(x)$ satisfying system (2.11), (2.12).

EXAMPLE 4.4. Consider Problem (VOP) with $n = p = 2$ and $m = 1$. For $x = (x_1, x_2) \in R^2$ define $f_1(x) = x_1 + x_2$, $f_2(x) = x_1 - x_2$ and $g_1(x) = x_1 + x_1x_2$. Observe that in our case (VOP) is KT-pseudoinvex at $x_0 := (0, 0) \in R^2$ and x_0 is not a weakly efficient point. Hence by Theorem 4.1 x_0 is not a generalized Kuhn–Tucker point.

As a consequence of Theorem 4.1 we obtain

THEOREM 4.1′. (a) *Every generalized Kuhn–Tucker point is a weakly efficient point of Problem (VOP) if and only if Problem (VOP) is KT-pseudoinvex.*

(b) *Under the closedness assumption, every Kuhn–Tucker point is a weakly efficient point of Problem (VOP) if and only if Problem (VOP) is KT-pseudoinvex.*

From Theorem 4.1′ we get the following corollary, the first statement of which is established in [11, 12] and the second one of which is a generalization of Theorem 2.1 of [9] to the nonsmooth case.

COROLLARY 4.1. (a) *Every Kuhn–Tucker point is a weakly efficient point of differentiable Problem (VOP) if and only if Problem (VOP) is KT-pseudoinvex.*

(b) *Every generalized Kuhn–Tucker point is a minimizer of Problem (P) (i.e., Problem (VOP) with $p = 1$) if and only if Problem (P) is KT-invex. Under the closedness assumption, every Kuhn–Tucker point is a minimizer of Problem (P) if and only if Problem (P) is KT-invex.*

Proof. Statement (a) is obtained from Theorem 4.1′ and Remark 4.1. Statement (b) is a consequence of Theorem 4.1′ and the remark that a weakly efficient point of Problem (P) is exactly its global minimizer and the KT-invexity coincides with the KT-pseudoinvexity (see Proposition 2.1). □

Before going further let us give a remark based on Theorem 4.1: to claim that a generalized Kuhn–Tucker point is a weakly efficient point we must check the KT-pseudoinvexity property of Problem (VOP). In some cases checking this property is a difficult task since we must detect a map η satisfying Definition 2.4. It is then natural to ask if we can find sufficient conditions for the KT-pseudoinvexity property without knowing η explicitly. Observe from Proposition 2.1 that HC-invexity and KT-invexity imply KT-pseudoinvexity. So, to answer the above question it is enough to give characterizations of HC-invexity and KT-invexity in terms of properties which are not related to map η mentioned in Definitions 2.2 and 2.3. The remainder of this section is devoted to these characterizations (see Theorems 4.2 and 4.3).

Let us set

$$S' = \{x \in S : f(x) - f(x_0) \neq 0\},$$
$$F(x_0, x) = \bigcup_{j \in J} [\partial f_j(x_0) \times \{f_j(x_0) - f_j(x)\}] \subset R^n \times R,$$
$$M(x_0, x) = F(x_0, x) \bigcup D_2(x_0, 0).$$

THEOREM 4.2. *Consider the following statements:*
(a) *Problem (VOP) is KT-invex on S at* x_0.
(b) *If* $\mu^l \geq 0,\ \lambda^l \geqq 0\ (l = 1, 2, \ldots)$ *and* $0 \in \lim\inf_{l\to\infty} H(\mu^l, \lambda^l, x_0)$ *then for all* $x \in S$

$$\lim\inf_{l\to\infty}[\langle\mu^l, f(x)\rangle - \langle\mu^l, f(x_0)\rangle] \geqq 0.$$

(c) *If* $\mu \geq 0,\ \lambda \geqq 0$ and $0 \in H(\mu, \lambda, x_0)$ then for all $x \in S$

$$\langle\mu, f(x)\rangle \geqq \langle\mu, f(x_0)\rangle .$$

Then
(a) ⇔ (b) ⇒ (c);
(a) ⇔ (b) ⇔ (c) *if for all* $x \in S'$ *the set* cone co $M(x_0, x)$ *is closed.*

We omit the proof of this theorem, observing that it is established on the basis of Proposition 3.2 and a modification of the proof of Theorems 4.3 and 4.3′ to be given later.

REMARK 4.4. If in the definition of KT-invexity we replace (2.11) by the following stronger condition

$$f(x) - f(x_0) \geqslant f^0(x_0, \eta(x))$$

then Theorem 4.2 (more exactly, implications (b) ⇒ (a) and (c) ⇒ (a) in Theorem 4.2) may fail to hold. The simplest counterexample is the case where $f_j \equiv 1\ (i = 1, 2, \ldots, p)$ and $g_i \equiv 0\ (i = 1, 2, \ldots, m)$.

Before turning to the HC-invexity property let us set

$$G(x_0, x) = \bigcup_{i\in I_0}[\partial g_i(x_0) \times \{g_i(x_0) - g_i(x)\}] \subset R^n \times R,$$

$$Q(x_0, x) = F(x_0, x) \bigcup G(x_0, x),$$

$$h(\mu, \lambda, x) = \sum_{j\in J} \mu_j f_j(x) + \sum_{i\in I_0} \lambda_i g_i(x).$$

Recall that $H(\mu, \lambda, x_0)$ is defined by (2.7).

THEOREM 4.3. *Consider the following statements*:
(a) *Problem (VOP) is HC-invex on S at* x_0.
(b) *If* $\mu^l \geqq 0,\ \lambda^l \geqq 0\ (l = 1, 2, \ldots)$ *and* $0 \in \lim\inf_{l\to\infty} H(\mu^l, \lambda^l, x_0)$ *then for all* $x \in S$

$$\lim\inf_{l\to\infty}[h(\mu^l, \lambda^l, x) - h(\mu^l, \lambda^l, x_0)] \geqq 0. \tag{4.11}$$

(c) *If* $\mu^l \geq 0,\ \lambda^l \geqq 0\ (l = 1, 2, \ldots)$ *and* $0 \in \lim\inf_{l\to\infty} H(\mu^l, \lambda^l, x_0)$ *then for all* $x \in S$ (4.11) *is satisfied.*
Then
(a) ⇔ (b) ⇒ (c);

(a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) *if condition (CQ) holds.*

Proof. (a) $\Rightarrow$ (b) Let $\mu^l \geqq 0$ and $\lambda^l \geqq 0$ be sequences such that
$0 \in \liminf_{l\to\infty} H(\mu^l, \lambda^l, x_0)$.
Then there are sequences $c_j^l \in \partial f_j(x_0)$ $(j \in J)$ and $b_i^l \in \partial g_i(x_0)$ $(i \in I_0)$ such that

$$0 = \lim_{l\to\infty} q^l, \tag{4.12}$$

where

$$q^l := \sum_{j\in J} \mu_j^l c_j^l + \sum_{i\in I_0} \lambda_i^l b_i^l. \tag{4.13}$$

Let $x \in S$ and let $\eta(x)$ be the point appearing in the definition of HC-invexity. Then we have from (4.13)

$$\langle q^l, \eta(x)\rangle \leqq h(\mu^l, \lambda^l, x) - h(\mu^l, \lambda^l, x_0). \tag{4.14}$$

Taking lim inf of both sides of (4.14) and using (4.12) we obtain (4.11), as required.

(b) $\Rightarrow$ (a) Let $x \in S$. We claim that

$$(0, 1) \notin \text{cl cone co } Q(x_0, x), \tag{4.15}$$

where 0 stands for the origin of R^n. Indeed, otherwise there are sequences

$$\mu^l \geqq 0, \quad \lambda^l \geqq 0, \quad c_j^l \in \partial f_j(x_0) \quad (j \in J), \quad b_i^l \in \partial g_i(x_0) \quad (i \in I_0) \tag{4.16}$$

such that (4.12) is satisfied and

$$1 = \lim_{l\to\infty} \left[h(\mu^l, \lambda^l, x_0) - h(\mu^l, \lambda^l, x)\right]. \tag{4.17}$$

This is impossible since by statement (b) the right side of the last equality is a nonpositive number. Applying Proposition 3.2 yields a point $\eta(x)$ satisfying (2.9), (2.10).

(b) $\Rightarrow$ (c) Obviously.

(c) $\Rightarrow$ (a) (under condition (CQ)). Let $x \in S$. As in the proof of implication (b) $\Rightarrow$ (a) it suffices to show the validity of (4.15). Assume to the contrary that (4.15) fails to hold. Then there are sequences (4.16) such that (4.12) and (4.17) are satisfied. We claim that $\mu^l \neq 0$ for l sufficiently large. Indeed, otherwise we have from (4.12) and (4.17), by taking a subsequence if necessary, that

$$0 = \lim_{l\to\infty} \sum_{i\in I_0} \lambda_i^l b_i^l, \tag{4.18}$$

$$1 = \lim_{l\to\infty} \sum_{i\in I_0} \lambda_i^l [g_i(x_0) - g_i(x)]. \tag{4.19}$$

From (4.19) it follows that $\lambda^l \neq 0$ for l sufficiently large. So $\gamma^l := \sum_{i\in I_0} \lambda_i^l \neq 0$.

Setting $\bar{\lambda}_i^l = \lambda_i^l/\gamma^l,\ b^l = \sum_{i\in I_0} \bar{\lambda}_i^l b_i^l$ we have from (4.18)

$$0 = \lim_{l\to\infty} \gamma^l b^l. \tag{4.20}$$

Since b^l belongs to the set co $\bigcup_{i\in I_0} \partial g_i(x_0)$ which is a compact set not containing the origin of R^n we may assume, by taking a subsequence if necessary, that b^l converges to some point $b \neq 0$. Therefore, making use of (4.20) we get

$$\lim_{l\to\infty} \gamma^l = \lim_{l\to\infty} \frac{\|\gamma^l b^l\|}{\|b^l\|} = \frac{0}{\|b\|} = 0.$$

This contradicts (4.19). We have thus proved that $\mu^l \geqslant 0$ for l sufficiently large. But in this case, by statement (c) (4.11) must be satisfied. This contradicts (4.17). □

THEOREM 4.3′. *Consider the following statements*:

(a)′ *Problem* (*VOP*) *is HC-invex on S at* x_0.

(b)′ *If* $\mu \geqq 0,\ \lambda \geqq 0$ *and* $0 \in H(\mu, \lambda, x_0)$ then for all $x \in S$

$$h(\mu, \lambda, x) \geqq h(\mu, \lambda, x_0). \tag{4.21}$$

(c)′ *If* $\mu \geqslant 0,\ \lambda \geqq 0$ *and* $0 \in H(\mu, \lambda, x_0)$ *then for all* $x \in S$ (4.21) *holds.*

Then

(a)′ ⇒ (b)′ ⇒ (c)′;

(a)′ ⇔ (b)′ *if for all* $x \in S\backslash\{x_0\}$ *the set* cone co $Q(x_0, x)$ *is closed*;

(a)′ ⇔ (b)′ ⇔ (c)′ *if condition (CQ) holds and if for all* $x \in S \setminus \{x_0\}$ *the set* cone co $Q(x_0, x)$ *is closed.*

Proof. (a)′ ⇒ (b)′ Use implication (a) ⇒ (b) of Theorem 4.3 and observe that (b) ⇒ (b)′.

(b)′ ⇒ (a)′ (under the extra assumption). Let $x \in S$. If $x = x_0$ then (2.9) and (2.10) are satisfied, with $\eta(x) = 0$. In case $x \neq x_0$ it suffices to show the validity of (4.15) which in our case means that $(0, 1) \notin$ cone co $Q(x_0, x)$. Indeed, otherwise there are $\mu \geqq 0$ and $\lambda \geqq 0$ such that

$$0 \in H(\mu, \lambda, x_0),$$

$$1 = -[h(\mu, \lambda, x) - h(\mu, \lambda, x_0)].$$

The last equality contradicts (4.21).

(b)′ ⇒ (c)′ Obviously.

(c)′ ⇒ (a)′ (under the extra assumptions). The proof is similar to that of implication (b)′ ⇒ (a)′. (Observe by condition (CQ) that the above vector μ must be different from zero.) □

REMARK 4.5. Results similar to those of Theorems 4.3 and 4.3′ are obtained in [5, p.8, 9] for a problem with constraints more general than (2.4). But they are valid only for the differentiable problem of scalar optimization while Theorem 4.3 and 4.3′ are established for *nonsmooth* problem of vector optimization.

5. Vector Optimization Problem on an Arbitrary Set

In this section we assume that S is an arbitrary nonempty subset of R^n which may not be given by inequalities (2.4). Consider the following *Vector Optimization Problem* (VOP)′:

$$\begin{aligned} &\min \quad f(x) \\ &\text{subject to} \quad x \in S, \end{aligned}$$

where $f = (f_1, f_2, \ldots, f_p)$ is a locally Lipschitz vector-valued map. Obviously, (VOP) is a special case of (VOP)′ with S being defined by (2.4). Problem (VOP)′ where f is Fréchet differentiable and S is open is considered in [11]. The case when $p = 1$ and f is nonsmooth is investigated in [14]. We shall see that the corresponding results of [11, Theorem 2.5] and [14, Theorem 4.1] are included in our Theorem 5.1 as special cases. We begin by the following definition.

DEFINITION 5.1. A point $x_0 \in S$ is a generalized stationary point of Problem (VOP)′ if

$$0 \in \text{ co } \bigcup_{j \in J} \partial f_j(x_0) + N_S(x_0). \tag{5.1}$$

DEFINITION 5.2. Problem (VOP)′ is KT-pseudoinvex on S at x_0 if there is a map $\eta : S \to T_S(x_0)$ such that

$$x \in S, \quad f(x) < f(x_0) \quad \Rightarrow \quad 0 > f^0(x_0, \eta(x)). \tag{5.2}$$

Problem (VOP)′ is KT-pseudoinvex if it is KT -pseudoinvex on S at any point $x_0 \in S$.

The notion of a *weakly efficient point* $x_0 \in S$ of Problem (VOP)′ is defined as in the case of Problem (VOP): for all $x \in S$ condition (2.5) does not hold.

We are now in a position to formulate

THEOREM 5.1. *Every generalized stationary point is a weakly efficient point of Problem* (*VOP*)′ *if and only if Problem* (*VOP*)′ *is KT-pseudoinvex.*

Proof. *Sufficiency*. Assume that Problem (VOP)′ is KT-pseudoinvex. We have to show that any point $x_0 \in S$ satisfying (5.1) must be a weakly efficient point. Indeed, otherwise there is $x \in S$ such that $f(x) < f(x_0)$. This implies by Definition 5.2 that (5.2) holds for a suitable point $\eta(x) \in T_S(x_0)$. Let $\xi \in \mathrm{co} \bigcup_{j\in J} \partial f_j(x_0)$ be such that $-\xi \in N_S(x_0)$ (see (5.1)). Then by (2.2) $\langle \xi, \eta(x) \rangle \geqq 0$. This contradicts (5.2).

Necessity. Assume that any generalized stationary point is a weakly efficient point of (VOP)′. Take an arbitrary point $x_0 \in S$. If x_0 satisfies (5.1) then Problem (VOP)′ is KT-pseudoinvex on S at x_0 since there is no $x \in S$ such that $f(x) < f(x_0)$. If (5.1) does not hold then by (2.2) this means that

$$0 \notin \mathrm{co} \bigcup_{j\in J} \partial f_j(x_0) + \mathrm{cl\ cone}\ \partial d_S(x_0).$$

Making use of Proposition 3.1 we get that system

$$f^0(x_0, \xi) < 0, \tag{5.3}$$

$$d_S^0(x_0, \xi) \leqq 0 \tag{5.4}$$

has a solution ξ. On the other hand, from the definition of d_S^0 we obtain that $d_S^0(x_0, \cdot) \geqq 0$. Combining this with (5.4) yields $d_S^0(x_0, \xi) = 0$ i.e. $\xi \in T_S(x_0)$. Thus by taking $\eta(x) = \xi \in T_S(x_0)$ for all $x \in S$ we obtain that $f^0(x_0, \eta(x)) < 0$. In other words, Problem (VOP)′ is KT-pseudoinvex on S at x_0. □

REMARK 5.1. If S is an open set then $T_S(x_0) = R^n$ and $N_S(x_0) = \{0\}$ for all $x_0 \in S$. From this it follows that Theorem 2.5 of [11] is included as a special case of our Theorem 5.1 with S being open.

DEFINITION 5.3. Problem (VOP)′ is KT-invex on S at x_0 if there is a map $\eta : S \to T_S(x_0)$ such that for all $x \in S$

$$f(x) - f(x_0) \geqq f^0(x_0, \eta(x)).$$

Problem (VOP)′ is KT-invex if it is KT-invex on S at any point $x_0 \in S$. Arguing as in the proof Proposition 2.1 we obtain

PROPOSITION 5.1. *For any Problem (VOP)′ and any point $x_0 \in S$*

KT-invexity on S at x_0 $\Rightarrow$ KT-pseudoinvexity on S at x_0.

The converse is true for Problem (P)′ (i.e. *Problem (VOP)′ with $p = 1$*).

As a direct consequence of Theorem 5.1 and Proposition 5.1 we get the following result of [14, Theorem 4.1] where the term "stationary point" is used instead of "generalized stationary point".

COROLLARY 5.1. *Every generalized stationary point of Problem $(P)'$ is a minimizer of $(P)'$ if and only if Problem $(P)'$ is KT-invex.*

We refer the reader to [13] for an invexity notion for non-Lipschitz functions which is introduced on the basis of a property of stationary points similar to that given in Corollary 5.1. In [13] stationary points are defined in terms of circa-subdifferentials which coincide with Clarke subdifferentials in Lipschitz case.

To conclude this section let us note that a result similar to Theorem 4.2 can be formulated for Problem (VOP)′ with the help of $\partial d_S(x_0)$.

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